Baccalauréat, toutes séries — Session de juin 2007

Épreuve de section européenne

Knuth's up-arrow notation

The multiplication $a \times b$ of a number a by a natural number b can be defined as an iteration of additions :

$$a \times b = \underbrace{a + a + \ldots + a}_{b \text{ copies of } a} .$$

In the same way, the exponentiation a^b of a number a to naturel number b is an iteration of multiplications :

$$a^b = \underbrace{a \times a \times \ldots \times a}_{b \text{ copies of } a} .$$

The expression a^b is sometimes noted $a \uparrow b$, as the standard notation cannot be used on a computer. In 1976, this notation inspired Donald Knuth, the famous mathematician and computer scientist, to create the notation $a \uparrow\uparrow$, defined as

$$a \uparrow\uparrow b = \underbrace{a \uparrow a \uparrow \ldots \uparrow a}_{b \text{ copies of } a} = \underbrace{a^{a^{\cdot^{\cdot^{a}}}}}_{b \text{ copies of } a} .$$

This notation is very convenient to write very large numbers.

Adapted from *Wikipedia.org*, the free encyclopedia.

Questions

1. Who introduced this notation, and when ?
2. Compute the numbers $3 \uparrow\uparrow 2$ and $3 \uparrow\uparrow 3$.
3. Try to compute the number $3 \uparrow\uparrow 4$ using your calculator. What goes on ?
4. Use this notation to write the number $A = 16$.
5. How many symbols do you need to write the number $B = 2^A$ using the decimal notation, the exponent notation and the up-arrow notation ?
6. The number $C = 2^B$ is close to $2.0 \times 10^{19,729}$. How many symbols do you need to write the number $B = 2^A$ using the decimal notation, the exponent notation and the up-arrow notation ? What notation is the most efficient in this case ?