Épreuve de section européenne

Invalid proofs

In mathematics, there are a variety of spurious proofs of obvious contradictions. Although the proofs are flawed, the errors, usually by design, are comparatively subtle. These fallacies are normally regarded as mere curiosities, but can be used to show the importance of rigor in mathematics. Here are a few examples :

Proof that 1 is less than 0

Let us first suppose that $0 < x < 1$. Now we will take the logarithm on both sides. As long as $x > 0$, we can do this because logarithms are monotonically increasing. Observing that the logarithm of 1 is 0, we get $\ln(x) < \ln(1)$, that is $\ln(x) < 0$ Dividing by $\ln(x)$ gives $1 < 0$.

Proof that 2 equals 1

Let a and b be equal non-zero quantities $a = b$. Multiply through by a. Subtract b^2 from both sides. Factor both sides. Divide out $(a - b)$. Observing that $a = b$, combine like terms on the left. Divide by the non-zero b. Conclude : $2 = 1$.

Proof that 4 equals 5

Start with the identity, $-20 = -20$. Express both sides in slightly different, yet equivalent ways

$$
\begin{aligned}
25 - 45 &= 16 - 36 \\
5^2 5 \times 9 &= 4^2 4 \times 9 \\
5^2 - 5 \times 9 + 9^2 &= 4^2 - 4 \times 9 + 9^2 \\
(5 - 9)^2 &= (4 - 9)^2 \\
5 - 9 &= 4 - 9 \\
-4 &= -5 \\
4 &= 5
\end{aligned}
$$

Adapted from *Wikipedia.org, the free encyclopedia*

Questions

1. In the first proof, explain the sentence "we can do this because logarithms are monotonically increasing". Where does the mistake come from in this proof ?

2. Complete the proof that shows that 2 equals 1. Why is this also an invalid proof ?

3. Explain each step of the proof that 4 equals 5, and then tell where the error comes from.