Épreuve de section européenne

Magical mathematics with integers.

Choose a three-digit number and write it twice in succession. For example, if you choose 761, then you should write down 761,761. The game begins by dividing your six-digit number by 7. The remainder, that is, whatever is left after the division, is your *lucky number.*

This lucky number will be one of the numbers 0, 1, 2, 3, 4, 5 or 6 ; now tell your lucky number to the teachers, and you will receive as many 100-euro notes as indicated by this lucky number !

If you are unfortunate enough to have ended up with zero as your lucky number, you are in good company, since the same fate will have befallen all of your fellow students : do you really believe the teachers would have accepted to spend their money this way ?

Those looking for a variant from dividing by 7 can substitute 11 or 13.

Can we try other lengths for the first chosen number ? With four-digit numbers we are dealing with multiplication by 10,001. This number is not prime, since $10,001 = 73 \times 137$, with both factors being prime. Therefore, if you write down a four-digit number twice to form an eight-digit number, it is guaranteed that it is divisible by 73 and 137. But who is eager to divide by 73 ?

Since the number 100,001 has only the prime divisors 11 and 9,091, both inconvenient divisors, five-digit numbers are not optimal for our magic trick. And so it goes. We again find small divisors with 1,000,000,001 (it is divisible by 7). But do we really want to begin our magic act with "choose a nine-digit number and write it down twice to form an eighteen-digit number" ? We recommend that you stick with the original trick.

Adapted from Ehrhard Behrends' *Five-Minute Mathematics*, AMS, 2008.

Questions

1. Choose another three-digit number and play the same game with it (including division in the English manner). Do you find a remainder different from zero ?
2. Why is the remainder one of the numbers 0, 1, 2, 3, 4, 5 or 6 ?
3. Choose a three-digit number and multiply it by 1,001. What is the result ? Can you explain it ?
4. Why is the number you get by writing twice the one you have chosen always divisible by 7 ? And what about substituting 11 or 13 to 7 in the game ?
5. About 1,000,000,001 : find the five prime factors of its decomposition [Hint : 52579 is a prime number.]