Épreuve de section européenne

The Kate Bush conjecture

In her song, Pi, Kate Bush sang the first one-hundred and fifty or so digits of the celebrity number. This is what she sang :

3.14159265358979323846264338327950288419716939937510582319749459230781640628620882148086513282306647093844609550582231

You don't need me to point out the wrong digits, do you ? Good. Then we can move on.

This has led to the Kate Bush Conjecture. Since Pi contains an infinite sequence of digits which never repeat, surely the sequence Kate sings must occur somewhere ! She never says she is starting at the beginning.

The Weak Kate Bush Conjecture says :

> The sequence Kate sings exists somewhere in the decimal expansion of Pi.

The Strong Kate Bush Conjecture says :

> Kate could have sung any finite sequence of digits and it would exist somewhere in the decimal expansion of Pi.

If Pi were a random sequence of digits then both conjectures are true. But Pi isn't random, it is a well-defined number so we can't make any assumptions. Instinctively I think it must be true, but that isn't good mathematics, we need to prove it !

For example, the following number is infinite and non-repeating but it doesn't satisfy either conjecture : $0.01001100011100001111\ldots$.

If the strong conjecture is true then every finite sequence exists in Pi. And they each exist an infinite number of times since they can occur in an infinite number of longer sequences. Think about that, an infinite number of sequences each occurs an infinite number of times.

Adapted from The Math Factor website, May 26, 2009.

Questions

1. What does the author imply about the digits sung by Kate Bush ?
2. Use your calculator to compute the first digits of π. Do you notice a mistake in the first digits sung by Kate Bush ?
3. The author says that π is not random, as it is a well-defined number. Give an accurate definition of π (it involves a circle).
4. Explain why the sequence given just before the last paragraph doesn't satisfy either conjecture. Is it random ?
5. **a.** Consider a randomly chosen sequence of three digits. What is the probability of this sequence being 123 ? Is it less or more than the probability of the sequence being 111 ?
   b. Consider a randomly chosen sequence of four digits. What is the probability of this sequence including the string 123 ?
   c. Consider a randomly chosen sequence of five digits. What is the probability of this sequence including the string 123 ?
   d. Consider a randomly chosen sequence of n digits, with n greater than 3. Conjecture a formula for the probability of this sequence including the string 123. try to prove it.