Épreuve de section européenne

The problem of the points

Consider a gambling game between two players in which at each stage one of the players gains a point. It might be something as simple as spinning a coin: if it is Heads, Player A gets a point, if it is Tails, Player B gets a point. The game is continued until one of the players reaches a fixed number of points and wins, taking all the stakes[1]. It might be 'first to 10' in which case Player A will win if the coin gives 10 Heads before it gives 10 Tails.

Suppose this game is interrupted, however. How should the stakes be divided? This is the 'problem of the points'. Obviously the player who is ahead should get more than the one behind, but in what ratio? One early solution was to divide the stakes in the ratio of the points already gained. This is very unfair if the score is $1-0$ because the player ahead will get all the stakes even though he has only a slender lead.

The Chevalier de Méré, a French aristocratic gambler, wrote to Pascal about the problem of the points. A correspondence then followed between Pascal and Fermat to discuss the problem. Fermat's solution, refined by Pascal, was to look to the future rather than the past. Instead of considering the points that had been played, they looked at the points that would be played if the game were allowed to continue. The stakes should be divided in the ratio of their probabilities of winning a continued game.

Consider the coin-spinning game mentioned at the beginning. Let us assume that two players, A and B, agree that the winner is the 'first to 10'. Suppose they reach a point in the game at which Player A has 9 Heads (H) and Player B has 8 Tails (T). There are up to two more spins left in the game. The possible results are as follows:

HH HT TH TT.

In three of these results Player A wins because 10 Heads are reached before 10 Tails. Only in the outcome TT does Player B win. Therefore the stakes should be divided in the ratio 3:1, so that Player A gets $\frac{3}{4}$ of the stakes.

From Robert Solomon, *The little book of mathematical principles*, NH, 2008.

Questions

1. Suppose we divide the stakes in the ratio of the points already gained (looking to the past) and that the game 'first to 10' is interrupted with 9 Heads for Player A and 8 Tails for Player B, as in the situation described in the text. Compute the ratio of the stakes for each player.
2. A game 'first to 10' is interrupted with 8 Heads for Player A and 8 Tails for Player B.
   (a) If the stakes are divided in the ratio of the points already gained, calculate the ratio of the stakes Player A obtains .
   (b) According to Pascal's method, compute the ratio of the stakes Player A gets.
3. A game 'first to 10' is interrupted with 7 Heads for Player A and 8 Tails for Player B.
   (a) Work out the maximum number of spins to be done if the game were continued until one of the two players gets 10 points.
   (b) Compute the ratio of the stakes gained by Player A using Pascal's solution.

[1] stakes = enjeux.