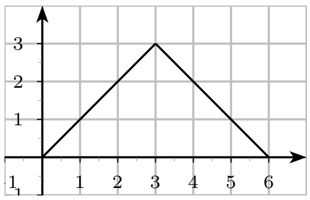
Épreuve de section européenne

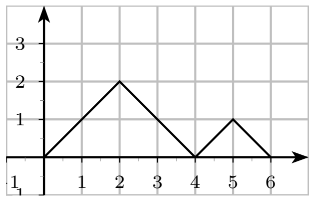
Catalan numbers

A Dick path of length $2n$ is a path that can be drawn with n upstrokes and n downstrokes. In other words, it's the number of different paths we can choose from the origin to the point $(2n, 0)$ on the xy-plane subject to the following conditions :

- We can touch the x-axis, but we cannot cross it.
- From any point (x, y), we can only climb up to the point $(x + 1, y + 1)$ or climb down to the point $(x + 1, y - 1)$.

Here are two Dick paths of length 6, which means $n = 3$.





Let's denote c_n the number of Dick paths of length $2n$. The numbers c_n are called Catalan numbers. There are many counting problems in combinatorics that involve the Catalan numbers.

From various sources

Questions

1. Give the two Dick paths of length 4 ($n = 2$). Use the empty grids below if you need.
2. Give the three other Dick paths of length 6 ($n = 3$). Use the empty grids below if you need.
3. It can be proved that the Catalan numbers satisfy the recurrence relation

$$c_0 = 1 \quad \text{and } c_{n+1} = \sum_{i=0}^{i=n} c_i c_{n-i} = c_0 c_n + c_1 c_{n-1} + c_2 c_{n-2} + \cdots + c_n c_0$$

So the expressions of the first Catalan numbers are as follows : $c_1 = c_0 c_0$, $c_2 = c_1 c_0 + c_0 c_1$ and $c_3 = c_2 c_0 + c_1 c_1 + c_0 c_2$. Give the formula for c_4 and compute it.

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