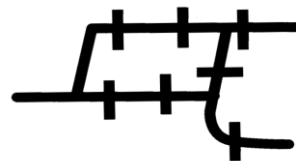## Épreuve de section européenne
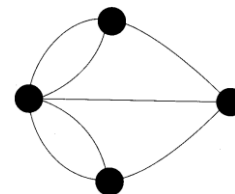
The Seven Bridges of Königsberg

In the 18th century the East Prussian city of Königsberg (now Kaliningrad, in Russia) had seven bridges crossing the river Preger. A simplified map of the city is shown beside.

Was it possible to tour the city, crossing each bridge once and once only? This was a popular puzzle of the time.
If a route could be found, it would have been found quickly. The mathematics to show that no such route is possible did not exist at the time, until Euler invented it.

Change the map, so that each region of land is represented by a dot and each bridge is represented by a line joining the dots. The leftmost dot represents the central island, which has five bridges leading to it.

The problem now becomes to draw the diagram without lifting the pen from paper and without going over any line twice.

The key to the solution is to count the number of lines leaving each dot.
At the beginning of the journey one can leave a dot without returning to it. At the end of the journey one can reach a dot without leaving it. But consider a middle stage of the journey. Every time one reaches a dot one must leave it. However many times the dot is reached, it must also be left.

So the number of lines from such a middle dot must be even. At most two of the dots (the beginning of the journey and its end) can have an odd number of lines from it.
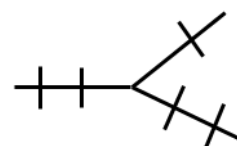
In the Königsberg diagram all four of the dots have an odd number of lines leaving from them. Therefore, it is impossible to draw the diagram continuously and without retracing a line, and so it is impossible to walk around Königsberg crossing each bridge once and only once.

Euler's solution became the basis for the foundation of graph theory.

From *The Little Book of Mathematical Principles*, by Robert Solomon

Questions

1. Give a definition of an even number and of an odd number.

2. Here is a map of a city, where two rivers meet, with five bridges joining the three areas of this city.
   a. Transform this map into a diagram with dots and lines as in the text.
   b. Is it possible to draw this diagram without lifting the pen from paper and without going over any line twice?
   c. Find a way to tour this city, crossing each bridge once and once only.
   d. Is it possible to tour the city, starting from an area and returning to it at the end of the journey?

3. Baby Euler has just learned to walk. He is curious to know if he can walk through every doorway in his house exactly once, and return to the room he started in.
   a. Will baby Euler succeed?
   b. Can baby Euler walk through every door exactly once and return to a different place from where he started?
   c. What if the front door is closed?